

Long rainbow path in properly edge-colored complete graphs*

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Abstract

Let G be an edge-colored graph. A rainbow (heterochromatic, or multicolored) path of G is such a path in which no two edges have the same color. Let the color degree of a vertex v be the number of different colors that are used on the edges incident to v , and denote it to be $d^c(v)$. It was shown that if $d^c(v) \geq k$ for every vertex v of G , then G has a rainbow path of length at least $\min\{\lceil \frac{2k+1}{3} \rceil, k-1\}$. In the present paper, we consider the properly edge-colored complete graph K_n only and improve the lower bound of the length of the longest rainbow path by showing that if $n \geq 20$, there must have a rainbow path of length no less than $\frac{3}{4}n - \frac{1}{4}\sqrt{\frac{n}{2} - \frac{39}{11} - \frac{11}{16}}$.

Keywords: properly edge-colored graph, complete graph, rainbow (heterochromatic, or multicolored) path.

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1. Introduction

We use Bondy and Murty [3] for terminology and notation not defined here and consider simple graphs only.

Let $G = (V, E)$ be a graph. By an *edge-coloring* of G we mean a function $C : E \rightarrow \mathbb{N}$, the set of natural numbers. If G is assigned such a coloring, then we

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say that G is an *edge-colored graph*. Denote the edge-colored graph by (G, C) , and call $C(e)$ the *color* of the edge $e \in E$. We say that $C(uv) = \emptyset$ if $uv \notin E(G)$ for $u, v \in V(G)$. For a subgraph H of G , we denote $C(H) = \{C(e) \mid e \in E(H)\}$ and $c(H) = |C(H)|$. For a vertex v of G , the *color neighborhood* $CN(v)$ of v is defined as the set $\{C(e) \mid e \text{ is incident with } v\}$, the *color degree* $d^c(v) = |CN(v)|$. A subgraph of G is called *rainbow* (*heterochromatic*, or *multicolored*) if any two edges of it have different colors. If u and v are two vertices on a path P , uPv denotes the segment of P from u to v , whereas $vP^{-1}u$ denotes the same segment but from v to u .

There are many existing publications dealing with the existence of paths and cycles with special properties in edge-colored graphs. The heterochromatic Hamiltonian cycle or path problem was studied by Hahn and Thomassen [14], Rödl and Winkler (see [11]), Frieze and Reed [11], and Albert, Frieze and Reed [1]. In [2], Axenovich, Jiang and Tuza gave the range of the maximum k such that there exists a k -good coloring of $E(K_n)$ that contains no properly colored copy of a path with fixed number of edges, no heterochromatic copy of a path with fixed number of edges, no properly colored copy of a cycle with fixed number of edges and no heterochromatic copy of a cycle with fixed number of edges, respectively. In [9], Erdős and Tuza studied the heterochromatic paths in infinite complete graph K_ω . In [10], Erdős and Tuza studied the values of k , such that every k -good coloring of K_n contains a heterochromatic copy of F where F is a given graph with e edges ($e < n/k$). In [15], Manoussakis, Spyratos and Tuza studied (s, t) -cycle in 2-edge colored graphs, where (s, t) -cycle is a cycle of length $s + t$ and s consecutive edges are in one color and the remaining t edges are in the other color. In [16], Manoussakis, Spyratos, Tuza and Voigt studied conditions on the minimum number k of colors, sufficient for the existence of given types (such as families of internally pairwise vertex-disjoint paths with common endpoints, hamiltonian paths and hamiltonian cycles, cycles with a given lower bound of their length, spanning trees, stars, and cliques) of properly edge-colored subgraphs in a k -edge colored complete graph. In [8], Chou, Manoussakis, Megalaki, Spyratos and Tuza showed that for a 2-edge-colored graph G and three specified vertices x, y and z , to decide whether there exists a color-alternating path from x to y passing through z is NP-complete. Many results in these papers were proved by using probabilistic methods.

In [2], Axenovich, Jiang and Tuza considered the local variation of anti-Ramsey problem, namely, they studied the maximum k such that there exists a k -good edge-coloring of K_n containing no heterochromatic copy of a given graph H , and denote it by $g(n, H)$. They showed that for a fixed integer $k \geq 2$, $k - 1 \leq g(n, P_{k+1}) \leq 2k - 3$, i.e., if K_n is edge-colored by a $(2k - 2)$ -good coloring, then there must exist a heterochromatic path P_{k+1} , and there exists an a $(k - 1)$ -good coloring of K_n such that no heterochromatic path P_{k+1} exists.

In [4], the authors considered long heterochromatic paths in general graphs with a k -good coloring and showed that if G is an edge-colored graph with $d^c(v) \geq k$ (color degree condition) for every vertex v of G , then G has a heterochromatic

path of length at least $\lceil \frac{k+1}{2} \rceil$. In [5, 6], we got some better bound of the length of longest heterochromatic paths in general graphs with a k -good coloring.

In [7], we showed that if $|CN(u) \cup CN(v)| \geq s$ (color neighborhood union condition) for every pair of vertices u and v of G , then G has a heterochromatic path of length at least $\lceil \frac{s+1}{2} \rceil$, and gave examples to show that the lower bound is best possible in some sense.

In [12], Gyárfás and Mhalla showed that in any properly edge-colored complete graph K_n , there is a rainbow path with no less than $(2n+1)/3$ vertices. In [6] we got a better result, showing that in any edge-colored graph G , if for every vertex of G there are at least k colors appear on it, then the longest rainbow path in G is no shorter than $\lceil \frac{2k}{3} \rceil + 1$.

Theorem 1.1 [6] *Let G be an edge-colored graph. If $d^c(v) \geq k$ for every vertex $v \in V(G)$, then G has a heterochromatic path of length at least $\min\{\lceil \frac{2k}{3} \rceil + 1, k - 1\}$.*

In this paper, we will improve the bound in [12], and show that a longest rainbow path in a properly edge-colored K_n is not shorter than $\left(\frac{3}{4} - o(1)\right)n$.

2. Propositions of a longest rainbow path

Suppose G is a properly edge-colored K_n , $P = v_0v_1v_2 \cdots v_l$ is one of the longest rainbow paths in G , and $C(v_{i-1}v_i) = C_i$ ($i = 1, 2, \dots, l$).

Suppose $l < n - 2$ and u is an arbitrary vertex which does not belong to the path P . Then we can easily get the following proposition.

Proposition 2.1 $C(v_0u) \in C(P)$, $C(v_lu) \in C(P)$.

Proof. Otherwise, uv_0Pv_l or $uv_lP^{-1}v_0$ is a rainbow path of length $l + 1$, a contradiction. ■

Proposition 2.2 If $C(uv_i) \notin C(P)$, then $C(uv_{i-1}) \in C(P)$, $C(uv_{i+1}) \in C(P)$.

Proof. Otherwise, $v_0Pv_{i-1}uv_iPu_l$ or $v_0Pv_iv_{i+1}Pu_l$ is a rainbow path of length $l + 1$, a contradiction. ■

Proposition 2.3 If $C(uv_i) \notin C(P)$, then $\{C(v_0v_{i+1}), C(v_lv_{i-1})\} \subset C(P) \cup C(uv_i)$.

Proof. Otherwise, $uv_iP^{-1}v_0v_{i+1}Pv_l$ or $uv_iPv_lv_{i-1}P^{-1}v_0$ is a rainbow path of length $l + 1$, a contradiction. ■

Proposition 2.4 *If $C(v_i v_l) \notin C(P)$, then $C(v_0 v_l) \in C(P) \setminus \{C_{i-1}, C_i\}$.*

Proof. Otherwise, $uv_i P v_l v_0 P v_{i-1}$ or $uv_i P^{-1} v_0 v_l P^{-1} v_{i+1}$ is a rainbow path of length $l + 1$, a contradiction. ■

Proposition 2.5 *If $C(v_0 v_i) \notin C(P)$, then $C(v_l u) \in C(P) \setminus C(v_{i-1} v_i)$; if $C(v_l v_i) \notin C(P)$, then $C(v_0 u) \in C(P) \setminus C(v_i v_{i+1})$.*

Proof. Otherwise, $v_{i-1} P^{-1} v_0 v_i P v_l u$ or $v_{i+1} P v_l v_i P^{-1} v_0 u$ is a rainbow of length $l + 1$, a contradiction. ■

Proposition 2.6 *If $C(v_0 v_i) \notin C(P)$, then $C(v_{i-1} u) \in C(P) \cup C(v_0 v_i)$; if $C(v_l v_i) \notin C(P)$, then $C(v_{i+1} u) \in C(P) \cup C(v_l v_i)$.*

Proof. Otherwise, $uv_{i-1} P^{-1} v_0 v_i P v_l$ or $uv_{i+1} P v_l v_i P^{-1} v_0$ is a rainbow of length $l + 1$, a contradiction. ■

With these propositions, we can give new lower bound of a longest rainbow path. And we will do that separately in the following two situations: the biggest rainbow cycle is of length $l + 1$, and the biggest rainbow cycle is of length less than $l + 1$.

3. A longest rainbow path has the same number of vertices as a biggest rainbow cycle

If the longest rainbow path has the same number of vertices as the biggest rainbow cycle, then the biggest rainbow cycle is of length $l + 1$, and there exists a rainbow path $P = v_0 v_1 \cdots v_l$ such that $C(v_0 v_l) \notin C(P)$.

Then, we can easily get the following conclusion from Proposition 2.4.

Lemma 3.1 *If $C(v_0 v_l) \notin C(P)$, then for an arbitrary $u \in V(G \setminus P)$, $C(u, P) \in C(P) \cup C(v_0 v_l)$.*

By using this Lemma, we can get one of our main conclusions.

Theorem 3.2 *If $n \geq 20$ and $C(v_0 v_l) \notin C(P)$, then $l \geq \frac{3}{4}n - 1$.*

Proof. We will prove it by contradiction. Suppose a longest rainbow path in G is of length $l < \frac{3}{4}n - 1$. Then $|V(G) \setminus V(P)| = n - l - 1 > \frac{n}{4} \geq 5$.

We can conclude by Lemma 3.1 that for any vertex $u \in V(G) \setminus V(P)$, $C(u, P) \subseteq C(P) \cup C(v_0v_l)$. On the other hand, $|V(P)| = |C(P) \cup C(v_0v_l)| = l+1$ and G is a properly edge-colored K_n . Therefore, $C(u, P) = C(P) \cup C(v_0v_l)$, $C(G \setminus P) \cap (C(P) \cup C(v_0v_l)) = \emptyset$.

Since P is one of the longest rainbow paths, by Proposition 2.1, there exist $2 \leq i_1 < i_2 < \dots < i_{n-2-l} < l$, $1 \leq j_1 < j_2 < \dots < j_{n-2-l} < l-1$, such that

$$\begin{aligned} & |\{C(v_0v_{i_1}), C(v_0v_{i_2}), \dots, C(v_0v_{i_{n-2-l}})\} \setminus (C(P) \cup \{C(v_0v_l)\})| \\ &= |\{C(v_lv_{j_1}), C(v_lv_{j_2}), \dots, C(v_lv_{j_{n-2-l}})\} \setminus (C(P) \cup \{C(v_0v_l)\})| \\ &= n-l-2 \end{aligned}$$

Additionally, $C(uv_{i_k-1}) \neq C(v_0v_l)$, $C(uv_{j_k+1}) \neq C(v_0v_l)$, $k = 1, 2, \dots, n-l-2$.

Let $I = \{i-1 \mid C(v_0v_i) \notin C(P) \cup C(v_0v_l), 2 \leq i \leq l-1\}$, $J = \{j+1 \mid C(v_jv_l) \notin C(P) \cup C(v_0v_l), 1 \leq j \leq l-1\}$.

Now we distinguish the following two cases:

Case 1. $I \cap J \neq \emptyset$.

This implies that there exists some t in $I \cap J$, i.e.,

$$\{C(v_0v_{t+1}), C(v_lv_{t-1})\} \cap (C(P) \cup C(v_0v_l)) = \emptyset$$

Case 1.1. $C(v_0v_{t+1}) \neq C(v_lv_{t-1})$.

Since $n-l \geq 4$ and $C(u, P) = C(P) \cup C(v_0v_l)$, there are no less than 3 colors which is not in $C(P) \cup C(v_0v_l)$ such that they belong to the color set $C(u, V(G) \setminus V(P))$. Therefore, there exist $u_1, u_2 \in V(G) \setminus V(P)$ such that $C(u_1u_2) \notin C(P) \cup \{C(v_0v_l), C(v_0v_{t+1}), C(v_{t-1}v_l)\}$.

By Lemma 3.1, there exists some vertex $v \in V(P)$ such that $C(u_1v) = C(v_0v_l)$, denote it by v_{i_0} . We can conclude from Proposition 2.6 that $i_0 \neq t$. Since $C' = v_0v_{t+1}Pv_lv_{t-1}P^{-1}v_0$ is a rainbow cycle of length l in which the color $C(v_0v_l)$ does not appear on it. Therefore, $u_2u_1v_{i_0}C$ contains a rainbow path of length $l+1$, a contradiction.

Case 1.2. $C(v_0v_{t+1}) = C(v_lv_{t-1}) \notin C(P) \cup C(v_0v_l)$.

First, we can conclude that $C(v_{t-1}u) \neq C(v_0v_l)$ for any vertex $u \in V(G) \setminus V(P)$. Otherwise, suppose there exists some $u \in V(G) \setminus V(P)$ such that $C(v_{t-1}u) = C(v_0v_l)$. Since $|V(G) \setminus V(P)| > 5$, there exists a vertex $u_1 \in V(G) \setminus (V(P) \cup \{u\})$ such that $C(uu_1) \notin C(P) \cup \{C(v_0v_l), C(v_0v_{t+1})\}$. Therefore, $u_1uv_{t-1}P^{-1}v_0v_{t+1}Pv_l$ is a rainbow path of length $l+1$, a contradiction.

Then, we will show that $t-1 \notin I \cup J$.

If $t-1 \in I$, i.e., $C(v_0v_t) \notin C(P) \cup \{C(v_0v_l), C(v_0v_{t+1})\}$, $C' = v_0Pv_{t-1}v_lP^{-1}v_tv_0$ is a rainbow cycle of length $l+1$ without color $C(v_0v_l)$. On the other hand, by 3.1 there exists a vertex $u \in V(G) \setminus V(P)$ and a vertex $v_{i_0} \in V(P)$ such that $C(uv_{i_0}) = C(v_0v_l)$. Then $uv_{i_0}C$ contains a rainbow path of length $l+1$, a contradiction.

If $t-1 \in J$, i.e., $C(v_{t-2}v_l) \notin C(P) \cup \{C(v_0v_l), C(v_{t-1}v_l)\}$, $C' = v_0Pv_{t-2}v_lP^{-1}v_{t+1}v_0$ is a rainbow cycle of length l without color $C(v_0v_l)$. Since $|V(G) \setminus V(P)| > 5$, for any vertex $u \in V(G) \setminus V(P)$, $d_{G \setminus P}^c(u) \geq 5$. So, by Theorem 1.1 there exists a rainbow path $u_1u_2u_3 \in G \setminus P$ with no colors in $C(P) \cup \{C(v_0v_l), C(v_0v_{t+1}), C(v_{t-2}v_l)\}$. Since G is properly edge-colored, at least one edge in $\{v_tu_1, v_tu_3\}$ does not have color $C(v_0v_l)$, W.O.L.G., assume $C(v_tu_1) \neq C(v_0v_l)$. Then, because $C(v_{t-1}u_1) \neq C(v_0v_l)$, $C(u_1, P) = C(P) \cup C(v_0v_l)$. So, by Lemma 3.1 there exists some i_0 , $0 \leq i_0 \leq l$, $i_0 \neq t-1, t$ such that $C(u_1v_{i_0}) = C(v_0v_l)$. Then $u_3u_2u_1v_{i_0}C'$ contains a rainbow path of length $l+1$, a contradiction.

So, we have $t-1 \notin I \cup J$.

Let $K = I \cap J$, $I' = (I \setminus K) \cup \{t-1 | t \in K\}$. Then $|I'| = |I|$ and $I' \cap J = \emptyset$.

Additionally, for any $t \in I' \cup J$ and any $u \in V(G) \setminus V(P)$, $C(v_tu) \neq C(v_0v_l)$. Otherwise, there exist some $t_0 \in K$ and some vertex $u \in V(G) \setminus V(P)$, such that $C(v_{t_0-1}u) = C(v_0v_l)$. Since $|V(G) \setminus V(P)| \geq 6$, there exists some vertex $u_1 \in V(G) \setminus V(P)$ such that $C(uu_1) \notin C(P) \cup \{C(v_0v_l), C(v_0v_{t_0+1})\}$. Then $u_1uv_{t_0-1}P^{-1}v_0v_{t_0+1}Pv_l$ is a rainbow path of length $l+1$, a contradiction.

On the other hand,

$$|I' \cup J| = |I'| + |J| = |I| + |J| \geq 2[(n-1) - (l+1)] = 2(n-l-2),$$

and $|V(G) \setminus V(P)| = n - (l+1) = n - l - 1$. So there are at least $n - l - 1$ i 's ($1 \leq i \leq l-1$) such that $C(uv_i) = C(v_0v_l)$ for some $u \in V(G) \setminus V(P)$. So we have $|I' \cup J| + (n-l-1) \leq l-1$, and then $2(n-l-2) + n-l-1 \leq l-1$, which implies $l \geq \frac{3}{4}n - 1$, a contradiction.

Case 2. $I \cap J = \emptyset$.

By Proposition 2.6, we have that for any $t \in I \cup J$ and any $u \in V(G) \setminus V(P)$, $C(v_tu) \neq C(v_0v_l)$. On the other hand, there are at least $|V(G) \setminus V(P)| = n-l-1$ i 's ($1 \leq i \leq l-1$) such that $C(uv_i) = C(v_0v_l)$ for some $u \in V(G) \setminus V(P)$. So we have $|I \cup J| + (n-l-1) \leq l-1$, and then $2(n-l-2) + n-l-1 \leq l-1$, which implies $l \geq \frac{3}{4}n - 1$, a contradiction.

This complete the proof. ■

4. A biggest rainbow cycle has less vertices than a longest rainbow path

Since a biggest rainbow cycle have less vertices than a longest rainbow path, then $C(v_0v_l) \in C(P)$.

For any longest rainbow path P , by Proposition 2.1 and Theorem 3.2, there

exist $2 \leq i_1 < i_2 < \dots < i_{t_1} < l$ ($t_1 \geq n - 1 - l$) such that

$$|\{C(v_0v_{i_1}), C(v_0v_{i_2}), \dots, C(v_0v_{i_{t_1}})\}| = |CN(v_0) \setminus C(P)| = t_1.$$

Now we will distinguish two cases: the case when there is a vertex $u \in V(G) \setminus V(P)$ such that $C(v_lu) = C_1$, and the case when there is no such vertex.

We first consider the case when there is a vertex $u \in V(G) \setminus V(P)$ such that $C(v_lu) = C_1$.

Theorem 4.1 *If $C(v_0v_l) \in C(P)$ and there is a vertex $u \in V(G)$ such that $C(v_lu) = C_1$, then $l \geq \frac{3}{4}n - \frac{1}{4}\sqrt{\frac{n}{2} - \frac{39}{11}} - \frac{11}{16}$.*

Proof. Suppose P is a longest rainbow path that has the minimized t_1 .

We can conclude from Proposition 2.5 that $C_{i_k} \notin C(v_l, C(G) \setminus V(P))$, $k = 1, 2, \dots, t_1$.

Let $C^1 = \{C_{i_k} | k = 1, 2, \dots, t_1\}$, $C_j^0 = CN(v_{i_{j-1}}) \setminus (C(P) \cup C(v_0v_{i_j}))$. Let the color set C_j^1 , C_j^* ($j = 1, 2, \dots, t_1$) be defined by the following procedure.

For $j = 1$ to t_1 do

$$C_j^* = \emptyset,$$

for $s = 1$ to $i_j - 3$

$$\text{if } C(v_{i_{j-1}}v_s) \in C_j^0, \text{ let } C_j^* = C_j^* \cup \{C_{s+1}\};$$

for $s = i_{j+1}$ to $l - 1$

$$\text{if } C(v_{i_{j-1}}v_s) \in C_j^0, \text{ let } C_j^* = C_j^* \cup \{C_s\},$$

$$C_j^1 = C_{j-1}^1 \cup C_j^*$$

Then we can conclude that $|C_j^*| = |C_j^0| \geq t_1 - 1$ by Proposition 2.1.

Suppose $|C_{t_1}^1| - |C^0| = j_0$ and $j \geq j_0 + 2$.

Let $C_{j,1} = \{C(v_{i_{t-1}}v_{i_t}) | t > j \text{ and } C(v_{i_{j-1}}v_{i_t}) \in C_j^0\}$,

$$C_{j,2} = \{C(v_{i_{t-1}}v_{i_t}) | t < j \text{ and } C(v_{i_{j-1}}v_{i_{t-1}}) = C(v_0v_{i_t})\},$$

$$C_{j,3} = \{C(v_{i_{t-1}}) | t < j \text{ and } C(v_{i_{j-1}}v_{i_{t-1}}) \in C_j^1 \setminus C(v_0v_{i_t})\}.$$

Then $C_{j,1}$, $C_{j,2}$, $C_{j,3}$ are mutually independent and $C_j^* \cap C^0 = C_{j,1} \cup C_{j,2} \cup C_{j,3}$. By the definition $|C_{j,1}| \leq t_1 - j$, $\bigcup_{j=j_0+2}^{t_1} C_{j,2} \subseteq \{C_{i_1}, C_{i_2}, \dots, C_{i_{t_1-1}}\}$ and $C_{j,2} \cap C_{j',2} = \emptyset$ since G is properly edge-colored.

Since $C(v_lu) = C_1$, we have $C_{j,3} = \emptyset$; otherwise, $v_2Pv_{i_{t-1}}v_{i_{j-1}}P^{-1}v_{i_t}v_0v_{i_j}Pv_lu$ is a rainbow path of length $l + 1$, a contradiction. Therefore, $C_j^* \cap C^0 = C_{j,1} \cup C_{j,2}$.

On the other hand, $|C_j^* \setminus C^0| \leq |C_j^1 \setminus C^0| \leq j_0$. So, $|C_{j,2}| = |C_j^* \cap C^0| - |C_{j,1}| \geq (t_1 - 1 - j_0) - (t_1 - j) = j - j_0 - 1$. Notice that $\sum_{j=j_0+2}^{t_1} |C_{j,2}| = \left| \bigcup_{j=j_0+2}^{t_1} C_{j,2} \right| \leq t_1 - 1$.

Then, we have $\sum_{j=j_0+2}^{t_1} (j - j_0 - 1) \leq t_1 - 1$, i.e., $\frac{1}{2} (t_1^2 - 2j_0 t_1 - t_1 + j_0^2 + j_0) \leq t_1 - 1$.

Therefore, $j_0 \geq t_1 - \frac{1}{2} - \sqrt{2t_1 - \frac{7}{4}}$, $|C_{t_1}| = t_1 + j_0 \geq 2t_1 - \frac{1}{2} - \sqrt{2t_1 - \frac{7}{4}}$.

Since $C(v_l, V(G) \setminus V(P)) \subseteq C(P) \setminus (C_{t_1}^1 \cup \{C_l\})$ and G is properly edge-colored, $|V(G) \setminus V(P)| \leq l - \left(2t_1 - \frac{1}{2} - \sqrt{2t_1 - \frac{7}{4}}\right) - 1$, i.e., $n - (l + 1) \leq l - \left(2t_1 - \frac{1}{2} - \sqrt{2t_1 - \frac{7}{4}}\right) - 1$. So, $2t_1 - \sqrt{2t_1 - \frac{7}{4}} \leq 2l - n + \frac{1}{2}$. Since $f(x) = 2x - \sqrt{2x - \frac{7}{4}}$ increases when $x > 2$ and $t_1 \geq n - l - 1 > 2$, we have

$$2(n - l - 1) - \sqrt{2(n - l - 1) - \frac{7}{4}} \leq 2l - n + \frac{1}{2}.$$

Therefore, $l \geq \frac{3}{4}n - \frac{1}{4}\sqrt{\frac{n}{2} - \frac{39}{16}} - \frac{11}{16}$.

This completes the proof. \blacksquare

Now we consider the case when for any longest rainbow path $P = v_0 v_1 v_2 \cdots v_l$ and any $u \in V(G) \setminus V(P)$, $C(v_l u) \neq C_1$.

Lemma 4.2 *If for any longest rainbow path $P = v_0 v_1 v_2 \cdots v_l$ and any $u \in V(G) \setminus V(P)$, $C(v_l u) \neq C_1$ and there are at most two j 's satisfying $2 \leq j \leq t_1$, $i_j - i_{j-1} \geq 2$, then $l \geq \frac{3n-4}{4}$.*

Proof. For any j ($1 \leq j \leq t_1$), $v_{i_{j-1}} P^{-1} v_0 v_{i_j} P v_l$ is a rainbow path. So we can get by Proposition 2.5 and the condition of this lemma that $\{C_{i_{j-1}}, C_{i_j}\} \cap C(v_l, V(G) \setminus V(P)) = \emptyset$.

Let $C^* = \bigcup_{j=1}^{t_1} \{C_{i_{j-1}}, C_{i_j}\}$. Then $|C^*| \geq 2t_1 - 2$ since there are at most two j 's satisfying $2 \leq j \leq t_1$, $i_j - i_{j-1} \geq 2$. On the other side, $C(v_l, V(G) \setminus V(P)) \subseteq C(P) \setminus (C^* \cup \{C_l\})$. So we have

$$n - l - 1 \leq l - (2t_1 - 2) - 1 = l - 2t_1 + 1 \leq l - 2(n - l - 1) + 1.$$

This implies that $l \geq \frac{3n-4}{4}$ and completes the proof. \blacksquare

Then we can get the following conclusion.

Theorem 4.3 *If $C(v_0v_l) \in C(P)$ and for any vertex $u \in V(G)$, $C(v_lu) \neq C_1$, then $l \geq \frac{3}{4}n - \frac{1}{4}\sqrt{\frac{n}{2} - \frac{39}{11}} - \frac{11}{16}$.*

Proof. Let $i_0 = \min\{i | \exists u \notin V(P) \text{ s.t. } C(v_lu) = C_i\}$. Suppose P is one of the longest rainbow paths such that i_0 is the smallest.

Let $j^* = \max\{j | i_j - i_{j-1} = 1\}$. Then we have $i_0 > i_{j^*}$; otherwise, $v_1Pv_{i_{j^*}-1}v_0v_{i_{j^*}}Pv_l$ is also a rainbow path of length l , but C_{i_0} appears on the $(i_0 - 1)$ -th edge of the path, a contradiction.

Now we distinguish the following two cases.

Case 1. $i_0 < i_{t_1}$.

Let the integer j_0 and the color sets $C_j^0, C_j^*, C_{j,1}, C_{j,2}, C_{j,3}$ be defined as in Theorem 4.1.

Suppose $i_{j_1-1} < i_0 < i_{j_1}$. Then we have that for any $j_1 \leq j_2 \leq t_1$, $\{C(v_{j_2-1}v_{i_t-1}) | 1 \leq t < j_1\} \cap C_{j_2}^0 = \emptyset$. Otherwise, there exists $j_3 < j_1 \leq j_2$, such that $C(v_{i_{j_3}-1}v_{i_{j_2}-1}) \notin C_{j_2}^0$. Then, $v_{i_{j_3}}Pv_{i_{j_2}-1}v_{i_{j_3}-1}P^{-1}v_0v_{i_{j_2}}Pv_l$ is a rainbow path of length l , but the color C_{i_0} appears on the $(i_0 - i_{j_3})$ -th edge of this path, a contradiction to the choice of P .

If there exists $j_1 \leq j_2 < j_3$ such that $C(v_{i_{j_3}-1}v_{i_{j_2}-1}) \notin \{C_{j_3}^0 \cup C(v_0v_{i_{j_2}})\}$, then $v_1Pv_{i_{j_2}-1}v_{i_{j_3}-1}P^{-1}v_{i_{j_2}}v_0v_{i_{j_3}}Pv_l$ is a rainbow path of length l , but C_{i_0} appears on the $(i_0 - 1)$ -th edge of this path, a contradiction.

Therefore, for any $j \geq j_1$, $C_{j,3} = \emptyset$, $C_{j,2} \subseteq \{C_{i_t} | j_1 \leq t < t_1\}$.

Case 1.1. $j_1 > j_0$.

As in Theorem 4.1, we can get that $\sum_{j=j_1}^{t_1} (j - j_0 - 1) = \sum_{j=j_1}^{t_1} |C_{j,2}| = \left| \bigcup_{j=j_1}^{t_1} C_{j,2} \right| \leq t_1 - j_1$. This implies that $(t_1 - j_1 + 1)(j_1 + t_1 - 2j_0 - 2) \leq 2(t_1 - j_1)$. Therefore, $j_0 \geq \frac{1}{2}[(t_1^2 - 3t_1) - (j_1^2 - 3j_1) + 2j_1 - 2] > \frac{1}{2}(2j_1 - 2) = j_1 - 1$, a contradiction.

Case 1.2. $j_1 \geq j_0$.

By the same calculation we did in Theorem 4.1, we can conclude that $l \geq \frac{3}{4}n - \frac{1}{4}\sqrt{\frac{n}{2} - \frac{39}{11}} - \frac{11}{16}$.

Case 2. $i_0 > i_{t_1}$.

If there are at most two j 's satisfying $2 \leq j \leq t_1$, $i_j - i_{j-1} \geq 2$, then by Lemma 4.2, $l \geq \frac{3n-4}{4} \geq \frac{3}{4}n - \frac{1}{4}\sqrt{\frac{n}{2} - \frac{39}{11}} - \frac{11}{16}$.

So we will only consider the case when there are at least three j 's satisfying $2 \leq j \leq t_1$, $i_j - i_{j-1} \geq 2$. Then $l \geq \frac{3n-4}{4}$. Suppose there are exactly k ($k \geq 3$) such j 's satisfying $s_1 < s_2 < \dots < s_k$. Then for any integer p ($1 \leq p \leq k$)

$v_1 P v_{i_{sp-1}} v_0 v_{i_{sp}} P v_l$ is a rainbow path of length l . Therefore,

$$C(v_1, V(G) \setminus V(P)) \subseteq (C(P) \setminus \{C_{i_{sp}}\}) \cup \{C(v_0 v_{i_{sp-1}}), C(v_0 v_{i_{sp}})\}.$$

Notice that $k \geq 3$, and so $\bigcap_{p=1}^k \{C(v_0 v_{i_{sp-1}}), C(v_0 v_{i_{sp}})\} = \emptyset$, and then $C(v_1, V(G) \setminus V(P)) \subseteq (C(P) \setminus \{C_{i_{sp}}\})$. Let $C^* = C(P) \setminus \left(\bigcup_{p=1}^s C_{i_{sp}} \cup \{C_1, C_2\} \right)$. Then $C(v_1, V(G) \setminus V(P)) \subset C^*$.

Case 2.1. $|C^* \cap \{C_1, C_2, \dots, C_{i_{t_1}}\}| < t_1$.

$|C^* \cap \{C_1, C_2, \dots, C_{i_{t_1}}\}| < t_1$ implies that $i_{t_1} - k - 2 < t_1$ and there exists a vertex $u \in V(G) \setminus V(P)$ such that $C(v_1 u) = C_t$, where $t \geq i_{t_1} - [t_1 - (i_{t_1} - k - 2)] = t_1 + k + 2$ and it appears on the $(l - t + 1)$ -th edge of the rainbow path $v_l P^{-1} v_{i_{s_1}} v_0 v_{i_{s_1-1}} P^{-1} v_1$ of length $l + 1$. By the choice of P , we can conclude that $l - t + 1 \geq i_0 > i_{t_1}$, i.e., $t \leq l - i_{t_1} + 1$. Remember that $i_{t_1} \geq 2t_1 - k$ and $t_1 \geq n - l - 1$, and so we have $t_1 + k + 2 \leq t \leq l - i_{t_1} + 1 \leq l - 2t_1 + k + 1$, i.e., $l \geq 3t_1 + 1 \geq 3n - 3l - 2$, and therefore $l \geq \frac{3n-2}{4} \geq \frac{3}{4}n - \frac{1}{4}\sqrt{\frac{n}{2} - \frac{39}{11}} - \frac{11}{16}$.

Case 2.2. $|C^* \cap \{C_1, C_2, \dots, C_{i_{t_1}}\}| \geq t_1$.

Suppose C_t is the t_1 -th color in C^* , $i_{j_0-1} < t \leq i_{j_0}$ and there are k_1 j 's in the set $\{2, \dots, j_0 - 1\}$ satisfying $i_j - i_{j-1} = 1$. Then we can conclude that $t = t_1 + k_1 + 2$ and $t > 2(j_0 - 1) - k_1 - 2 = 2j_0 - k_1 - 4$. Since if $i_p - i_{p-1} > 1$ then $|C^* \cap \{C_{i_{p-1}+1}, \dots, C_{i_p}\}| \leq i_p - i_{p-1}$, we have

$$t_1 = i_1 - 2 + \sum_{\substack{p \leq j_0-1 \\ i_p - i_{p-1} > 1}} (i_p - i_{p-1}) + t - i_{j_0-1} = t_1 - k_1 - 2.$$

On the other hand, $i_{t_1} \geq i_{j_0} + 2(t_1 - j_0) - (k - k_1) = i_{j_0} + 2t_1 - 2j_0 - k + k_1 \geq t + 2t_1 - 2j_0 - k + k_1$. By Lemma 4.2 there is some integer j satisfying $i_j - i_{j-1} = 1$, and so $v_l P^{-1} v_{i_j} v_0 v_{i_{j-1}} P^{-1} v_1$ is a rainbow path of length $l + 1$ and C_t appears on the $(l - t)$ -th or $(l - t + 1)$ -th edge. Therefore, we have $i_0 \leq l - t$ by the choice of P . Then we have $l - t \geq i_0 > i_{t_1} \geq t + 2t_1 - 2j_0 - k + k_1$, i.e.,

$$\begin{aligned} l - t_1 - k - 2 &\geq 3t_1 + 2k_1 - 2j_0 + 2 \\ &> 3t_1 + 2k_1 + (-t - k_1 - 4) + 2 \\ &= 3t_1 - t + k_1 - 2 \\ &= 3t_1 - (t_1 + k_1 + 2) + k_1 - 2 \\ &= 2t_1 - 4 \end{aligned}$$

So, $l \geq 3t_1 + k - 2 \geq 3t_1 - 2 \geq 3(n - l - 1) - 2$, which implies that

$$l \geq \frac{3n-5}{4} \geq \frac{3}{4}n - \frac{1}{4}\sqrt{\frac{n}{2} - \frac{39}{11}} - \frac{11}{16}.$$

This completes the proof. ■

5. Conclusion

By Theorems 3.2, 4.1 and 4.3, we can easily get the following conclusions.

Theorem 5.1 *For any properly edge-colored complete graph K_n ($n \geq 20$), there is a rainbow path of length no less than $\frac{3}{4}n - \frac{1}{4}\sqrt{\frac{n}{2} - \frac{39}{11} - \frac{11}{16}}$.*

Corollary 5.2 *For any properly edge-colored complete graph K_n ($n \geq 20$), there is a rainbow path of length no less than $(\frac{3}{4} - o(1))n$.*

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